125:305 Numerical Methods in Biomedical Engineering

9/21/17 Homework 2

Due in class on (Tuesday) 10/3/17 – Late submissions will not be graded

You are free to discuss the questions with other students, but *EVERYONE MUST SUBMIT THEIR OWN WORK*

Question 1: [7 points]

Stryker produces three different knee replacement implants ("Triathlon", "Restoris", and "Tritanium"), each with different numbers of components inside.
https://www.stryker.com/content/stryker/us/en/portfolios/orthopaedics/joint-replacement/knee.html

The table below shows the number of each component (pins, screws, and hinges) required to build each model.

	Component		
Model name:	**Pins**	**Screws**	**Hinges**
Triathlon	4	4	2
Restoris	6	2	1
Tritanium	5	3	3

In one week, 1110 pins, 690 screws, and 480 hinges are shipped to the company.

(a) Write down a system of equations that model the production run in both conventional and matrix form [4 points]

(b) Write a Matlab script to calculate how many of each prosthetic knee model the company can produce that week. Compare the solutions obtained for two direct solution methods (use built-in Matlab functions). [2 points]

(c) How long does your code take to find a solution when each of these Matlab direct solution methods are used? [1 points]

Question 2 [12 points]:

UNITED STATES PATENT OFFICE

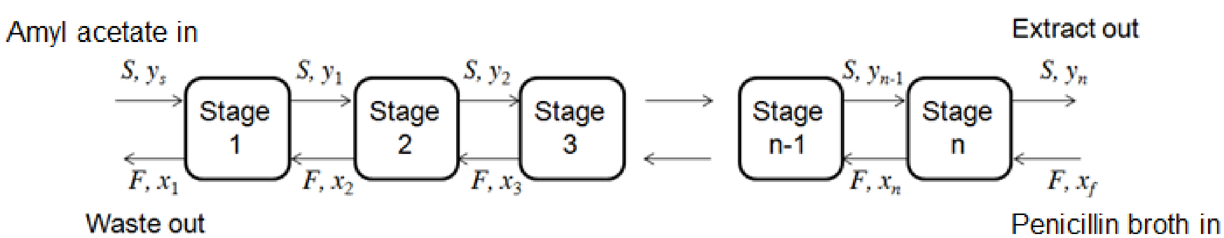
2,563,779

RECOVERY OF PENICILLIN FROM
FERMENTATION BROTH

Robert K. Finn, Westfield, N. J., assignor to
Merck & Co., Inc., Rahway, N. J., a corporation
of New Jersey

The process to extract purified penicillin from a fermentation broth uses a series of reaction chambers with the raw broth solution flowing in the opposite direction to a solvent extraction stream (amyl acetate). The goal is to separate the penicillin from the broth by extracting the drug in a pure solvent. In each reaction chamber, the broth and the solvent do not mix, but some of the penicillin diffuses into the solvent. The exiting solvent is called the extract.

In the extraction system shown below, the input flow rate of solvent (amyl acetate) is S cm^3/hr and of penicillin broth is F cm^3/hr. The concentration of penicillin entering stage "n" in the feed stream is x_{n+1} g/cm^3 and the concentration of penicillin entering stage "n" in the extraction stream is y_{n-1} g/cm^3.



(a) [2 points] Assume that the mass flow rates (S, F) don't change over the length of the reaction system. Use steady state mass balance (mass in per hr = mass out per hr) to write an equation for each stage when we have $n = 6$ stages.

(b) [3 points] Use the distribution coefficient $K = y_i / x_i$ in each stage to eliminate the y_i term in each of your equations in part (a). Express these equations in matrix form, [A][x] = [b].

What physical part(s) of the system is represented by the [b] vector?

Hint: Gather terms in "x" on the left side of your equations in (a) in order x_1, x_2, x_3, … and move constant terms to the right side. Matrix [A] will contain combinations of S, F, and K. Column vector [x] will contain the unknown x_1, x_2, x_3,…

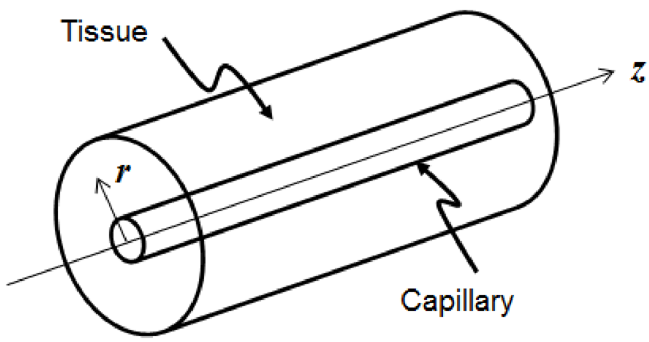
(c) [2 points] Use one of Matlab's direct algorithms to determine the concentration of penicillin entering each stage in the feed stream when S = 125 cm^3/hr, F = 265 cm^3/hr, x_f = 0.028 g/cm^3, y_s = 0 g/cm^3, and K = 2. Plot these concentrations as a function of stage #.

(d) [2 points] How many iterations does Matlab's iterative solution algorithm `lsqr` need to reach a solution with a relative residual (error) below 1 x 10^{-10}?

(d) [3 points] As the number of stages in the extraction system increases, more of the penicillin is transferred over to the extract stream. How many stages are needed to achieve an extract concentration above 0.051 g/cm^3 under the same conditions as (c)? Plot the concentration of penicillin in the extract stream as a function of stage # for this case.

Question 2: [10 points]

The *Krogh tissue cylinder* is a simplified model for evaluating transport of metabolites from the capillary vessels to the surrounding tissue. The Krogh model geometry is shown below. The capillary is modeled as a cylinder of constant radius (r_c), with wall thickness (t_m), oriented in length along the positive z-axis. As a solute moves through the capillary lumen, its concentration decreases due to radial diffusion of molecules through the capillary wall and into the surrounding tissue.



Property	**Value**
Capillary inner radius, r_c	0.0005 cm
Capillary wall thickness, t_m	5×10^{-5} cm
Average blood velocity, V	0.041 cm/s
Entering concentration, C_0	5 μm/cm³
Tissue glucose consumption rate, R_0	0.03 μm/cm³.s
Glucose tissue diffusivity, D_T	8×10^{-4} cm²/s
Mass transfer coefficient, K_0	5.75×10^{-5} cm/s

At some radial distance into the tissue (r_{crit}), the concentration of any metabolite will have reduced to zero. Solution of the Krogh tissue cylinder model leads to a nonlinear expression describing r_{crit} as a function of distance z along the capillary:

$$R^2\ln\left(R^2\right)-R^2+1-\left[\frac{4D_TC_0}{R_0\left(r_c+t_m\right)^2}\right]+4\frac{D_T}{Vr_c^2}\left[R^2-1\right]z+\frac{2D_T}{r_cK_0}\left[R^2-1\right]=0$$

where $R=\dfrac{r_{crit}}{r_c+t_m}$. This can be written in more compact form as:

$$R^2\ln\left(R^2\right)=A+E\left(R^2-1\right)$$

with $A=\dfrac{4D_TC_0}{R_0\left(r_c+t_m\right)^2}$ $\quad B=1-\dfrac{2D_T}{r_cK_0}$ $\quad D=\dfrac{4D_T}{Vr_c^2}$ $\quad E=\left(B-Dz\right)$

(i) Write a Matlab script to calculate the critical tissue radius (r_{crit}) where glucose is no longer supplied to cells, at a distance z = 0.025 cm under the conditions listed in the table. Use the Newton-Raphson method with ε_a = 0.1%. Generate plots showing your root estimates and approximate relative error at each iteration.

Hint: First find R^2, then use this to calculate r_{crit}.

[6 points]

(ii) Modify your Matlab script from (i) to generate a plot of r_{crit} versus z as z ranges from 10 μm to 0.15 cm.

[4 points]